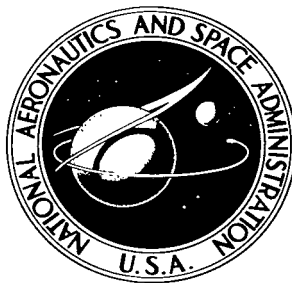


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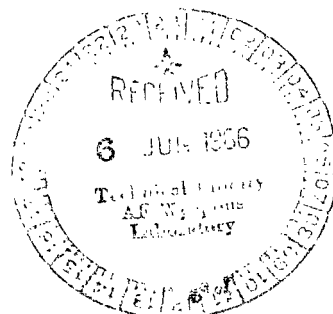


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# SIMPLIFIED CALCULATION OF TRANSITION MATRICES FOR OPTIMAL NAVIGATION

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Moffett Field, Calif.*





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FOR OPTIMAL NAVIGATION

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# SIMPLIFIED CALCULATION OF TRANSITION MATRICES FOR OPTIMAL NAVIGATION

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## SUMMARY

This paper presents the results of a study aimed at simplifying the equations used when applying the Kalman filter to space navigation. One frequently used method for generating the transition matrix is to integrate the perturbation equations of motion. An alternate approach considered here generates approximate matrices directly from the gravitational potential. With this scheme, the transition matrices can be generated much more quickly and with smaller computer storage requirements than before. Using these approximate matrices for navigation is equally as good a technique as the original.

## INTRODUCTION

In previous studies concerned with space navigation (refs. 1-6) the use of an optimal filter for processing the data has been considered. These studies have shown that such a filter permits accurate extraterrestrial navigation. Since the computations required are fairly complex, it would be desirable to see what simplifications could be made without too much loss of accuracy.

In this study we will consider the possibility of using simpler (and possibly less accurate) methods for computing the transition matrix. This matrix is used in space navigation calculations to relate deviations of the trajectory at one time to those at another. Thus,

$$x(t) = \phi(t; t_0)x(t_0)$$

where  $\phi(t; t_0)$  is the transition matrix from  $t_0$  to  $t$ , and  $x(t_0)$  and  $x(t)$  represent the state at two different times.

In the previous work the transition matrix has been obtained by integrating the perturbation equations of motion (ref. 1, appendixes B and E). This approach utilizes a reference trajectory (obtained by integration of the true nonlinear equations of motion) to compute the coefficients of the perturbation equations and gives a transition matrix whose accuracy is limited only by the accuracy of integration. The computational procedure, however, is quite complex.

Two possible means of simplifying this computation will be considered here. First, one can approximate the transition matrix by a power series in the two variables  $t$  and  $t_0$ , and then use this series to compute the  $\phi$  matrix; second, one can derive a simplified expression for the matrix which is a

function of the partials of the gravitational field and the time increment. For both cases the necessary equations will be derived and results presented and compared with the results obtained by using the true matrices, obtained by integrating the perturbation equation.

## ANALYSIS

In the previous studies (refs. 1-3) the transition matrix has been obtained by integration of the perturbation equations of motion. The differential equation governing the motions of the state variable  $x$  (composed of three position and three velocity components) is

$$\dot{x}(t) = F^*(t)x(t) \quad (1)$$

The  $F^*$  matrix can be partitioned into  $3 \times 3$  submatrices, as

$$F^* = \begin{bmatrix} 0 & I \\ F & 0 \end{bmatrix} \quad (2)$$

where  $F$  is the gradient of the gravitational attraction.

One of the properties of the transition matrix is that it satisfies the same differential equation as does  $x$ . Thus,

$$\dot{\phi}(t; t_0) = F^*(t)\phi(t; t_0); \quad \phi(t_0; t_0) = I \quad (3)$$

where double arguments are used to indicate the finishing and starting time. The solution of this expression can be expressed as a two-dimensional Taylor series in  $t_0$  and  $\delta t$ , where  $\delta t = t - t_0$  with the origin for  $t$  and  $t_0$  assumed to be at the start of the trajectory. We then have for each element of the transition matrix

$$\phi_{n,m}(t; t_0) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{ij} \frac{t_0^i}{i!} \frac{\delta t^j}{j!} \quad (4)$$

The  $C_{ij}$  are a set of appropriate coefficients that can be determined by a two-dimensional curve fitting process. That is, one can evaluate  $\phi_{n,m}(t; t_0)$  along some specific reference trajectory by integrating the perturbation equations and then evaluate a set of  $C_{ij}$  that will cause the series to converge to the previously determined value of  $\phi_{n,m}$ . This process must be repeated for all elements of the matrix  $\phi$ .

This approach has two drawbacks. First, the shape of the  $\phi$  surface is highly irregular as  $t_0$  approaches the time of periapse. This is demonstrated in figures 1(a) and (b), which are plots of two typical elements of  $\phi$ . These are plotted for a constant time increment,  $\delta t$ , and varying starting time,  $t_0$ .

In contrast, plots of these elements versus  $\delta t$  for fixed  $t_0$  (not shown) are quite smooth. Thus, for a good fit, the series must contain high powers of  $t_0$  and therefore a large number of terms. Fitting the surface for only the midrange values of  $t_0$  one would require many fewer terms. Then, however, some other technique would be needed to obtain transition matrices at the end points of  $t_0$ . The other difficulty is that the  $C_{ij}$  are evaluated along a specific reference trajectory and would have to be re-evaluated for different trajectories.

### Development of F Series

An alternate approach is to expand the solution of equation (3) in a one-dimensional Taylor series in  $\delta t$ , giving

$$\varphi(t; t_0) = \varphi(t_0; t_0) + \dot{\varphi}(t_0; t_0)\delta t + \ddot{\varphi}(t_0; t_0) \frac{\delta t^2}{2!} + \dots \quad (5)$$

Equations for the derivatives of  $\varphi$  can be obtained appropriately differentiating equation (3) and evaluating the results at  $t = t_0$ . The derivatives of  $\varphi$  are

$$\left. \begin{aligned} \varphi(t_0; t_0) &= I \\ \dot{\varphi}(t_0; t_0) &= F^*(t_0) \\ \ddot{\varphi}(t_0; t_0) &= \dot{F}^* + F^*F^* \\ \ddot{\ddot{\varphi}}(t_0; t_0) &= \ddot{F}^* + 2\dot{F}^*F^* + F^*\dot{F}^* + F^{*3} \\ &\text{etc.} \end{aligned} \right\} \quad (6)$$

If the partitioned form of  $F^*$  from equation (2) is substituted into equation (6) and these, in turn, are substituted into equation (5), the following expression for  $\varphi$  results

$$\begin{aligned} \varphi(t; t_0) &= I + \begin{bmatrix} 0 & I \\ F(t_0) & 0 \end{bmatrix} \delta t + \begin{bmatrix} F(t_0) & 0 \\ \dot{F}(t_0) & F(t_0) \end{bmatrix} \frac{\delta t^2}{2!} \\ &\quad + \begin{bmatrix} \dot{F}(t_0) & F(t_0) \\ \ddot{F}(t_0) + F(t_0)^2 & 2\dot{F}(t_0) \end{bmatrix} \frac{\delta t^3}{3!} + \dots \\ &= \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{bmatrix} \end{aligned} \quad (7)$$

where

$$\varphi_1 = I + F(t_0) \frac{\delta t^2}{2!} + \dot{F}(t_0) \frac{\delta t^3}{3!} + \dots$$

$$\varphi_2 = I\delta t + F(t_0) \frac{\delta t^3}{3!} + \dots$$

$$\varphi_3 = F(t_0)\delta t + \dot{F}(t_0) \frac{\delta t^2}{2!} + [\ddot{F}(t_0) + F(t_0)^2] \frac{\delta t^3}{3!}$$

$$\varphi_4 = I + F(t_0) \frac{\delta t^2}{2!} + 2\dot{F}(t_0) \frac{\delta t^3}{3!}$$

where again  $\delta t = t - t_0$ .

In this equation,  $F$  may be as complex or as simple as desired, that is,  $F$  may include only two-body effects, or may include additional terms due to oblateness and other bodies of the solar system. Also it is apparent that one may include more or fewer terms in the series evaluation. For simplicity, it would be desirable to use only the terms involving  $F$ , dropping terms involving powers and derivatives of  $F$ . This gives an estimate of the transition matrix as

$$\varphi_{\text{est}}(t; t_0) = \begin{bmatrix} I + F(t_0) \frac{\delta t^2}{2!} & I\delta t + F(t_0) \frac{\delta t^3}{3!} \\ F(t_0)\delta t & I + F(t_0) \frac{\delta t^2}{2!} \end{bmatrix} \quad (8)$$

Since  $F$  is a function of position in orbit, this equation can be evaluated easily for any trajectory and any time along the trajectory. Thus, no reference trajectory is required. Further, it is relatively simple, as discussed in the next section, to determine how big  $\delta t$  can be allowed to get.

#### Development of Maximum Allowable Time Interval

In using equation (8) there is some maximum value of  $\delta t$ , beyond which the neglected terms in the series become important. This value of  $\delta t$  can be determined by equating the first term dropped with the last term kept. Since the terms all involve matrices, it seems reasonable to use the norm of the matrix in the equation. This leads to the following four equations.

$$\|F(t_o)\| \frac{\delta t^2}{2!} = \|\dot{F}(t_o)\| \frac{\delta t^3}{3!}$$

$$\|F(t_o)\| \frac{\delta t^3}{3!} = \|2\dot{F}(t_o)\| \frac{\delta t^4}{4!}$$

$$\|F(t_o)\| \delta t = \|\dot{F}(t_o)\| \frac{\delta t^2}{2!}$$

$$\|F(t_o)\| \frac{\delta t^2}{2!} = 2\|\dot{F}(t_o)\| \frac{\delta t^3}{3!}$$

If these equations are solved for  $\delta t$ , one finds that the smallest  $\delta t$  is obtained from the last equation, and this value of  $\delta t$  is used to determine a  $\delta t_{\max}$ . That is, we wish to find a  $\delta t_{\max}$  such that

$$\|F(t_o)\| = \frac{2}{3} \|\dot{F}(t_o)\| \delta t_{\max} \quad (9)$$

and then in evaluating equation (8), we will choose  $\delta t \leq k \delta t_{\max}$ , where  $k$  is less than unity.

Let us define

$$\|F\| = \max_{i,j} |f_{ij}|$$

as being a reasonable norm to use in evaluating equation (9), and further, let us restrict  $F$  to the two-body case. We then have (ref. 6, eq. (6.51))

$$F = -\frac{\mu}{r^3} \left( I - \frac{3\bar{r}\bar{r}^T}{r^2} \right) \quad (10)$$

where  $\mu$  is the constant of gravitational attraction;  $\bar{r}^T = (r_1, r_2, r_3)$  is the position vector of the vehicle with respect to the gravitational center; and  $r = |\bar{r}|$ .

In order to determine  $\|F\|$ , we must choose that  $\bar{r}$  which will maximize the element of  $F$  being considered. There are really only two types of terms to consider - diagonal and off diagonal. A typical diagonal term is  $f_{11} = -(\mu/r^3)[1 - (3r_1^2/r^2)]$  which can be maximized if  $r_1 = r$  giving  $|f_{11\max}| = 2\mu/r^3$ . A typical off diagonal term is

$$f_{12} = \frac{3\mu}{r^3} \frac{r_1 r_2}{r^2}$$



which is a maximum when  $r_1 = r_2 = r/\sqrt{2}$ , giving  $|f_{12\max}| = (3/2)(\mu/r^3)$ . Thus, choosing the larger of these two values, we have

$$\|F\| = 2 \frac{\mu}{r^3} \quad (11)$$

Having determined  $\|F\|$  for use in equation (9), we will now consider  $\|\dot{F}\|$ . Differentiating equation (10) gives

$$\dot{F} = \frac{v}{r} \frac{3\mu}{r^3} \left[ \frac{\bar{r}\bar{v}^T}{rv} + \frac{\bar{v}\bar{r}^T}{rv} + \frac{\bar{r}^T\bar{v}}{rv} \left( I - 5 \frac{\bar{r}\bar{r}^T}{r^2} \right) \right] \quad (12)$$

where

$$\bar{v}^T = (v_1, v_2, v_3) = \frac{d\bar{r}}{dt}^T \quad \text{and} \quad v = |\bar{v}|$$

The evaluation of  $\|\dot{F}\|$  is slightly more complicated since both  $r$  and  $v$  must be considered, along with their scalar product. To simplify the evaluation of  $\|\dot{F}\|$ , only values of  $\bar{r}^T\bar{v}/rv = 0, \pm 1$  were considered, and it was assumed that for other values of the product the maximum element would be nearly the same. Using the same general method as before, we get

$$\|\dot{F}\| = 6 \frac{v}{r} \frac{\mu}{r^3} \quad (13)$$

Substituting equations (13) and (11) into (9) and solving for  $\delta t_{\max}$  gives

$$\delta t_{\max} = \frac{r}{2v} \quad (14)$$

Thus, if  $\delta t = r/2v$ , the size of the terms neglected in equation (8) (those involving  $\dot{F}$ ) are comparable with those retained (involving  $F$ ). It is expected then that  $\delta t$  will be chosen such that  $\delta t \leq k\delta t_{\max}$  with  $k$  being 1.0 or less.

A check was also made on the size of the second neglected term (not given in these equations), and it was found that for  $\delta t = \delta t_{\max}$  this term at perigee is slightly less than the  $\dot{F}$  term and becomes relatively smaller as the distance to the central body increases. Also, since it involves an additional power of  $\delta t$ , it decreases rapidly as  $\delta t$  is reduced from  $\delta t_{\max}$ .

In many cases, the desired  $\delta t$  (say from one observation to another or from the beginning to the first observation) is larger than  $k\delta t_{\max}$ , and thus two or more transition matrices must be combined to give the desired overall transition matrix. For example, suppose we desire the matrix  $\phi(t_1; t_0)$ . However,  $k\delta t_{\max}(t_0) = \delta t_a < t_1 - t_0$ . We can readily generate  $\phi(t_a; t_0)$ , where  $t_a = t_0 + \delta t_a$ , from equation (8). Then consider the time interval from  $t_a$  to  $t_1$ . Assuming that now  $k\delta t_{\max}(t_a) > t_1 - t_a$ , we can obtain  $\phi(t_1; t_a)$ , and by multiplying we get

$$\varphi(t_1; t_0) = \varphi(t_1; t_a) \varphi(t_a; t_0) \quad (15)$$

Of course, this process may be repeated as many times as required to obtain the desired overall matrix.

### Effect of Two-Body Assumption

The previous evaluation of  $F$  involved only two-body terms. One should also consider additional effects, specifically, earth oblateness and the perturbing forces of the sun and moon, and should determine the importance of such effects relative to the neglected terms in the power series expansion of  $F$ . Including these effects gives

$$\begin{aligned} F = & -\frac{\mu_e}{r_e^3} \left( I - \frac{3\bar{r}_e\bar{r}_e^T}{r_e^2} \right) - \frac{\mu_m}{r_m^3} \left( I - 3 \frac{\bar{r}_m\bar{r}_m^T}{r_m^2} \right) - \frac{\mu_s}{r_s^3} \left( I - 3 \frac{\bar{r}_s\bar{r}_s^T}{r_s^2} \right) \\ & - \frac{\mu_e}{r_e^5} J a^2 \left[ (I + 2M) \left( I - \frac{5\bar{r}_e\bar{r}_e^T}{r_e^2} \right) \right. \\ & \left. - 5 \left( \frac{\bar{r}_e^T M \bar{r}_e}{r_e^2} I + \frac{2\bar{r}_e\bar{r}_e^T}{r_e^2} M - \frac{7\bar{r}_e\bar{r}_e^T}{r_e^2} M \frac{\bar{r}_e\bar{r}_e^T}{r_e^2} \right) \right] \end{aligned} \quad (16)$$

where

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (17)$$

It should be noted that this is the matrix form of the perturbation equations, which can be obtained by partial differentiation of the equations in appendix A of reference 1. We will consider the norms of each of the terms in (16) individually and consider their relative effects. The first three terms have the same form, and the form of the norm has been already given in equation (11). Thus,

$$\|F_e\| = 2 \frac{\mu_e}{r_e^3}$$

$$\|F_m\| = 2 \frac{\mu_m}{r_m^3}$$

$$\|F_s\| = 2 \frac{\mu_s}{r_s^3}$$

The last term of equation (16) reaches its maximum if  $\vec{r}$  is all in the  $z$  direction with the (3,3) term being the largest. This gives approximately

$$\|F_J\| = \frac{\mu_e}{r_e^5} J a^2 [3(-4) - 5(1 + 2 - 7)] = 8 \frac{\mu_e}{r_e^5} J a^2$$

These four components of  $\|F\|$  are evaluated at four points along the earth-moon line to get representative values. The points used are at the earth's surface, at 66,000 km from the earth, 66,000 km from the moon (the sphere of influence), and at the moon's surface. The distance from the sun was assumed constant at 1 a.u., and the earth-moon distance was assumed to be 380,000 km.

The results are tabulated in table I. As expected, in the vicinity of the earth the  $\|F_e\|$  dominates, while in the vicinity of the moon the  $\|F_m\|$  dominates. Thus, if only a single two-body term is to be used for  $F$ , it must be switched between  $F_e$  and  $F_m$  in the vicinity of the sphere of influence, or alternatively, both terms can be used at all times. The importance of these various terms will be discussed in the results section where, having determined a value for  $k$ , we will be able to estimate the approximate size of the higher order terms of the series that have been rejected.

## DESCRIPTION OF SIMULATION

In order to consider the effects of using approximate transition matrices, a digital computer program was developed. This program is a modification of the one used by McLean (ref. 2). The principal modifications were made to the navigation system equations. These equations are discussed in the following subsection, and are followed by a description of the trajectory used in the simulation. Other factors in the simulation are as used by McLean (ref. 2) and also by Smith (ref. 7).

### Navigation System Equations

When an approximate transition matrix is used in connection with the equations for the optimal filter, the overall system becomes suboptimal. As a result, the error covariance matrix obtained is only an approximation and is not a proper measure of the true estimation error. A true measure - that is, a true covariance matrix of estimation error - requires additional equations. The optimal filter equations have been developed previously (refs. 1 and 2),<sup>1</sup> as well as the equations giving the statistics of suboptimal filters (ref. 7). The necessary equations will be repeated here for convenience. It is assumed that observations are uncorrelated from one time to the next.

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<sup>1</sup>Obviously, the Kalman filter technique as applied to the navigation problem is not optimal, strictly speaking, since the theory assumes a linear system and the navigation system equations are only approximately linear. However, for the purposes of this report it is assumed that the linear approximation is exact and the covariance matrix of estimation error computed by means of the Kalman filter equation is, therefore, a true measure of system performance.

Consider first the true covariance matrix of estimation errors. This matrix can be updated from the time of one observation to the time of the next by

$$P^-(t_{k+1}) = \phi(t_{k+1}; t_k) P^+(t_k) \phi^T(t_{k+1}; t_k) \quad (18)$$

where the  $-$  and  $+$  superscripts refer to before and after an observation, respectively;  $P^+(t_0)$  is given as an initial condition; and  $\phi(t_{k+1}; t_k)$  is obtained by integrating the perturbation equations of motion.

The estimated state  $\hat{x}$  is also integrated from  $t_k$  to  $t_{k+1}$ . At the time of an observation the estimated trajectory is updated using an estimated weighting matrix as

$$\hat{x}^+ = \hat{x}^- + K_{est}(y - H\hat{x}^-) \quad (19)$$

Here  $y$  represents the observation and  $H$  is the matrix of partial differentials relating deviations in the trajectory to deviations in the observation. Thus  $H\hat{x}^-$  represents the expected value of the observation based on previous data;  $K_{est}$  is a weighting matrix which is expected to be an estimate of the optimum weighting matrix.

As a result of this observation, the true covariance matrix  $P$  is updated as

$$P^+ = (I - K_{est}H)P^-(I - K_{est}H)^T + K_{est}QK_{est}^T \quad (20)$$

This equation is valid for a nonoptimal system (see ref. 7, eq. (72)) and gives the covariance matrix associated with the state estimation obtained in equation (19) for any  $K_{est}$ .

Equations (18) and (20) could not be used in an actual system that used approximate transition matrices, since the true transition matrices would not be available. For the purpose of this study, however, these equations were used to define the true state of affairs.

The actual system will use the equations for a true optimal system, but will use an estimate of the transition matrix,  $\phi_{est}(t_{k+1}; t_k)$  computed by means of the F-series (or any other desired method). These equations will then produce only an estimate of the covariance matrix and weighting matrix.

The procedure starts by updating the estimated covariance matrix  $P_{est}$  by means of equation (18), but using  $\phi_{est}$  and  $P_{est}$ , where  $P_{est}^+(t_0) = P^+(t_0)$ . The estimated state  $\hat{x}$  is updated to the appropriate time by integration and equation (19) is used exactly as given to determine the new estimate. The weighting matrix required in equation (19) to update the estimated trajectory is given as

$$K_{est} = P_{est}^- H^T (H P_{est}^- H^T + Q)^{-1} \quad (21)$$

and  $P_{est}^+$  is given as

$$P_{est}^+ = P_{est}^- - K_{est} H P_{est}^- \quad (22)$$

If the true covariance matrix  $P$  is used in equation (21), the resulting  $K_{est}$  will be optimum. If this optimum  $K_{est}$  is then substituted into equation (20), this equation will then simplify and have the same form as equation (22).

In order to get an indication of the nonoptimality caused by the use of the approximate transition matrix, the true matrix,  $P$ , is compared with the estimated matrix,  $P_{est}$ . To make this comparison, the quantities  $P_r$  and  $P_v$  will be determined. These are defined as follows: The covariance matrix is partitioned into four  $3 \times 3$  submatrices. The square root of the trace of the upper left submatrix is  $P_r$  and that of the lower right is  $P_v$ . Thus,  $P_r$  represents the rms position deviation of the estimated trajectory with respect to the actual trajectory while  $P_v$  represents the corresponding velocity deviations. Similar quantities from  $P_{est}$ , that is,  $P_{rest}$  and  $P_{vest}$ , are also determined.

### Trajectory

In order to test the navigation accuracy using an approximate calculation of the transition matrix, a sample circumlunar mission was chosen for a reference. The particular reference trajectory chosen has a 70.68 hour flight time to the moon with a perilune altitude of 187 km. It has a free return to the earth arriving at perigee at 144.87 hours after launch and entering the earth's atmosphere near the center of the entry corridor. The navigational observations used were theodolite measurements, which determine the right ascension and declination of the earth or moon as seen from the vehicle. Appropriately chosen sextant data would give essentially the same results. There were 45 observations made on the outbound leg of the mission and 39 on the return leg. The observations were grouped in a manner very similar to that shown in figure 2 of reference 7; that is, observations of the earth were grouped together, as were those of the moon, and some of both types were made shortly before each of the three expected velocity corrections on each leg.

### RESULTS AND DISCUSSION

The purpose of this section is to discuss the operation of the navigation system using approximate transition matrices and to consider the effects of the various simplifications. First of all, we will discuss the accuracy of the F-series transition matrices. Then we will consider the step size required for use with the F-series expansions, and will compare the resultant covariance matrices with those obtained using the exact transition matrix in a truly optimum system. The effect of gravitational perturbations is then considered followed by a brief discussion of computer storage and time requirements. Finally, the effect on guidance will be considered.

## Transition Matrices

In the computer program the true transition matrix, calculated by integrating the perturbation equations, was available, as well as the F-series approximation. These two matrices were compared (using the norms of the submatrices), and the percentage difference between the estimate and the true transition matrix varied from 1 to 0.1 percent for long intervals where many transition matrices must be multiplied together and from 0.1 to 0.001 percent for short intervals where a single matrix can be used. McGee (ref. 8) states that adequate navigation can be performed if the transition matrices for short intervals are calculated using only 16 bits or 0.003 percent error. The F-series transition matrix has about this error for the short intervals and thus one would expect that it would provide adequate navigation, as indeed it does.

## Effect of Step Size

In order to determine the effect of using various time intervals in calculating the transition matrix from the F-series, a comparison was made between  $P_{est}$ , the approximate covariance matrix of errors in the estimated trajectory, and  $P$ , the true covariance matrix. The F-series itself was calculated omitting the sun and oblateness terms, but including at all times both the earth and moon terms to provide a smooth transition from earth-centered to moon-centered conics.

A study of  $P_r$  and  $P_v$  from both the true and estimated covariance matrices will then show the adequacy of the transition matrix approximation. A plot of  $P_{rest}/P_r$  and  $P_{vest}/P_v$  for both outbound and return legs of the circumlunar trajectory is shown in figure 2 for four different values of  $k$ . Values of  $P$  and  $P_{est}$  were available only after an observation. The gaps in the curves represent periods of no observations. Ideally, of course, these ratios should be constant at unity.

It can be seen in figures 2(a) and (b) that, at least until perilune, a value of  $k = 0.1$  gives quite good results and that the accuracy falls off as  $k$  increases, so that at  $k = 1.0$  the results are poor. After perilune, however, all of these curves deviate from unity, and data not shown indicate that this trend continues so that at perigee the ratio is quite far from unity. However, if the run is restarted at perilune with  $P = P_{est} =$  nominal values, then as shown in figures 2(c) and (d), the results are very similar to those of the outbound leg, with  $k = 0.1$  quite good and  $k = 1.0$  fairly poor.

It is apparent, however, that smaller values of  $k$  are required for adequate results after passing perilune. If the F-series is calculated at every integration step, which implies that  $k$  is small, the ratio of  $P_{est}/P$  remains close to 1.0 for the complete round trip. Thus, these smaller values of  $k$  in the vicinity of the moon allow a successful perilune passage. However, this gives a considerable increase in the amount of computation. If the vehicle is returning from a lunar orbit, it would be natural to restart the  $P_{est}$  matrix at its nominal value so that extremely small values of  $k$  would

not be required. In a fly-by mode, however, this would not be natural. Nevertheless, at perilune a nominal  $P$  matrix could be inserted in the computer, which would allow successful computations during the return flight.

These results indicate that the assumptions made in determining  $\delta t_{\max}$  (eq. (14)) are reasonable, and that a value of  $k = 0.25$  gives reasonably accurate results. This value is used as a standard in the remainder of the discussion.

Using this value of  $k$ , one must multiply several short transition matrices together to cover some of the larger required intervals. For instance, 30 short term matrices were required to generate the first matrix from  $t = 0$  to 2.5 hours, and 15 were required during the last 48 hours to perilune. Most of the time a single matrix was adequate to cover the time interval between observations.

### Comparison With Optimum System

It is of interest to compare the results of using the approximate transition matrices with the results of using the true matrices in a truly optimum system. This comparison is made in figures 3 and 4. In figure 3, the true transition matrices and the optimal  $K$  matrices were used throughout. Thus  $P_{\text{est}}$  and  $P$  are equal and optimum. The curves represent  $P_r$  and  $P_v$ , and the points represent the navigation error of the estimated trajectory for a single typical flight. The navigation error is supposed to be a typical member of the ensemble for which  $P$  is the rms value. It can be seen that the points could easily represent a typical member.

In figure 4 the estimated transition matrix was used, and  $P_{\text{rest}}$  and  $P_{\text{vest}}$  are plotted along with the associated navigation error. The curves in figures 3 and 4 are similar with only small differences, indicating that  $P_{\text{est}}$  is a good measure of the optimum  $P$ . The navigation error points are also similar. This is to be expected since identical sequences of simulated observation errors were used in each case, and the covariance matrices are also similar.

### Effect of Two-Body Calculations

In the previous results, both the moon and earth gravitation terms were used at all times to compute the F-series. It might be simpler to use only one of these terms and to switch terms at the sphere of influence. These two techniques are compared for  $k = 0.25$  in figure 5. The plots of  $P_{\text{est}}/P$  for the two cases are essentially identical from launch to the sphere of influence of the moon. Following this, there is some difference, especially in figure 5(b), which tends to decrease with time. This same effect is also noticeable during the return trajectory. In general, either technique will work adequately, but the three-body technique gives slightly better results.

## Effect of Oblateness and Solar Gravitation

It is also of interest to ascertain the solar and oblateness effects. In table I the size of these terms is given along with those of the earth and moon. The sun's effect is always less than that of either the earth or the moon, and so will have considerably less influence than that shown in figure 5 for the difference between two- and three-body calculations. The oblateness term is largest at perigee, at this point, for  $k = 0.25$ , the second term in the series, which is being neglected, is about  $0.6(10)^4$ , or about 40 times larger than the oblateness term. Since the relative size of the oblateness term decreases rapidly on leaving perigee, it is apparent that it will have less effect than truncating the series. In conclusion, it appears that the solar gravitation and earth's oblateness will have very little effect on the overall results.

## Computer Storage and Time Comparison

Since the F-series calculation of the transition matrix appears to give reasonable results, it is of interest to determine the computer storage and computational time requirements for comparison with the corresponding requirements of the technique where the perturbation equations are directly integrated. Computer programs are available for both techniques, and a limited amount of data is available concerning storage and timing. However, both programs are written in Fortran IV, and no optimization of either has been attempted. Thus, the storage and time requirements given here must be considered approximate. For each of these two programs the storage specifically required for the calculation of the transition matrix and the time required for performing these calculations during the outbound leg of the mission were determined and are presented in the following table:

|                          | <u>Storage, words</u> | <u>Time, min</u>                            |
|--------------------------|-----------------------|---|
| F-series                 | 600                   | ( $k = 0.1$ ) $1/4$<br>( $k = 0.25$ ) $1/5$ |
| Perturbation integration | 1400                  | $3/4$                                       |

It can be seen that the F-series takes somewhat less than half the storage required for the perturbation integration technique. The time required to integrate the perturbation equation is constant at about  $3/4$  of a minute; whereas the time required for the F-series calculations is a function of  $k$ . In the extreme where the F-series is evaluated at every integration step, the time required is about  $3/4$  minute. Thus, the F-series approach results in savings in both storage and time.

## Effect on Vehicle Guidance

It has been shown that an approximate transition matrix can be used satisfactorily for navigation in cislunar space, that is, for determining the vehicle's present position and velocity. Once the navigation is completed, the results are used in one of several available guidance procedures to determine



what, if any, corrective maneuver is required so that the vehicle will satisfactorily complete the mission. Using the transition matrix for guidance as well as navigation has been studied in previous research (refs. 2, 6). In those studies it was assumed that the vehicle would be required to arrive at a desired specified point at a particular time. The transition matrix, from the present to the end point, was obtained (ref. 2) by multiplying many short individual matrices together. This guidance scheme was used in the present study in two ways: First, the approximate transition matrices were used for navigation only, and the true transition matrices were used in the guidance scheme. In this case, the standard deviation of the miss at arrival was about the same as previously reported, namely, about 2 km in altitude and 20 km in range. This indicates that the approximate transition matrix provides adequate navigation. Second, it was desired to see the effect of using the approximate transition matrices in the guidance section also. In a study of word length requirements (ref. 8), it was concluded that this guidance scheme would be usable with only 16-bit accuracy in the computation of the transition matrices. The approximate matrices used here have about 16-bit accuracy for the majority of the time, but at the beginning and at the end, the accuracy decreases to about 6 bits. When these approximate transition matrices were used in the guidance scheme, the standard deviation of the altitude miss was about 1000 km, indicating that these transition matrices are unsatisfactory for this purpose.

Thus, one can conclude that these approximate transition matrices are quite satisfactory for use in navigation, but that a guidance scheme should be selected which does not use them.

## CONCLUSIONS

This report shows that a simplified technique for computing the transition matrices is available which provides adequate navigation in cislunar space and which uses less computer storage and less computation time. These transition matrices are computed as a function of the gravitational attraction along the estimated trajectory and so do not require a reference trajectory.

The gravitational attraction can be computed using the standard two-body equations. The effect of the earth's oblateness and the sun can be neglected. In going to the moon, the moon's attraction can either be included all the time, or the gravitational center can be switched at the lunar sphere of influence.

The maximum time over which a single calculation of the gravitational attraction can be used to generate the transition matrices is easily calculated. Where a longer time interval is required, several individual transition matrices can be computed and then multiplied together to obtain the overall desired matrix.

Ames Research Center  
National Aeronautics and Space Administration  
Moffett Field, Calif., Feb. 28, 1966

## APPENDIX

### DEFINITION OF SYMBOLS

|           |  |
|-----------|--|
| a         | equatorial radius of the earth   |
| $C_{ij}$  | coefficient of the $ij$ th term of a two-dimensional power series  |
| F         | gradient of the gravitational attraction vector  |
| $F^*$     | $\begin{bmatrix} 0 & I \\ F & 0 \end{bmatrix}$ , the matrix of coefficients in the differential equation of motion of the state vector |
| $\ F\ $   | norm of the F matrix   |
| $f_{ij}$  | the $ij$ element of the F matrix   |
| H         | matrix of partial derivatives of the observed quantity with respect to the state variables   |
| I         | identity matrix of suitable dimension  |
| J         | coefficient of the second harmonic of the earth potential  |
| K         | optimal weighting matrix   |
| k         | constant used in determining allowable step sizes  |
| O         | null matrix of suitable dimension  |
| P         | covariance matrix of estimation errors   |
| $P_r$     | rms position deviation of the estimated trajectory with respect to the actual trajectory   |
| $P_v$     | rms velocity deviation of the estimated trajectory with respect to the actual trajectory   |
| Q         | covariance matrix of observation errors  |
| r         | $ \bar{r} $  |
| $\bar{r}$ | $(r_1, r_2, r_3)^T$ , position vector of the vehicle   |
| t         | time   |
| v         | $(v_1, v_2, v_3)^T$ , velocity vector of the vehicle   |

|   |   |
|---|---|
| $x$   | state vector representing position and velocity deviations from the reference |
| $\hat{x}$   | state vector of estimated deviation from the reference                        |
| $\delta t$  | time difference over which the transition matrix is used                      |
| $\delta t_{\max}$   | maximum allowable value for $\delta t$  |
| $\mu$   | constant of gravitational attraction  |
| $\varphi$   | transition matrix relating deviations at one time with those at another       |
| $\left. \begin{matrix} \varphi_1, \varphi_2, \\ \varphi_3, \varphi_4 \end{matrix} \right\}$ | submatrices of $\varphi$  |

#### Notational Conventions

|                       |                         |
|-----------------------|-------------------------|
| $(\dot{\phantom{x}})$ | time derivative of ( )  |
| $(\phantom{x})^T$     | transpose of matrix ( ) |
| $(\phantom{x})^{-1}$  | inverse of matrix ( )   |

#### Subscripts

|       |                   |
|-------|-------------------|
| e     | earth             |
| est   | estimated         |
| i,j,k | indices           |
| J     | earth oblateness  |
| m     | moon              |
| o     | initial condition |
| s     | sun               |

#### Superscripts

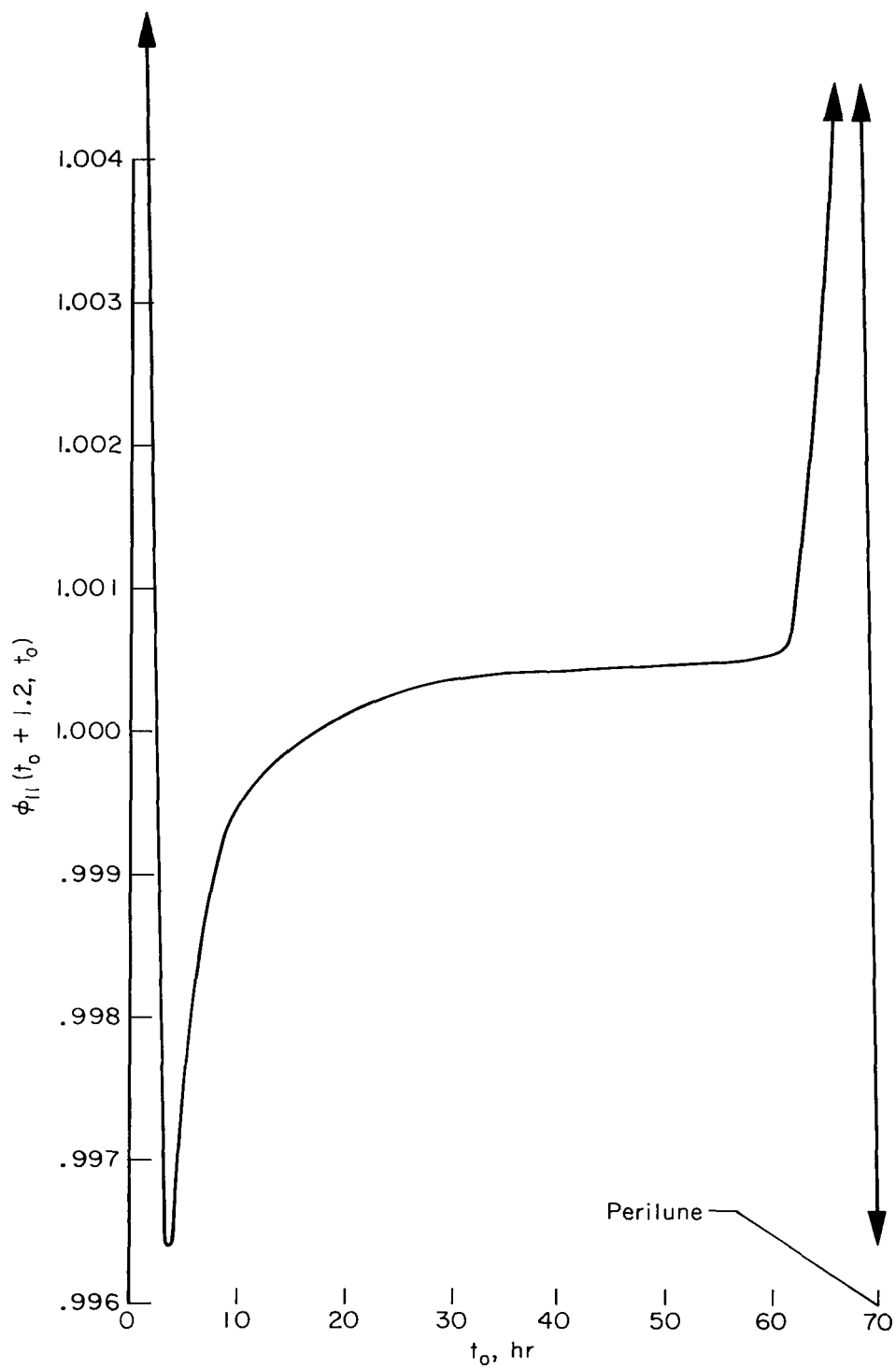
|   |                             |
|---|-----------------------------|
| - | value before an observation |
| + | value after an observation  |

## REFERENCES

1. Smith, Gerald, L.; Schmidt, Stanley F.; and McGee, Leonard A.: Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity On Board a Circumlunar Vehicle. NASA TR R-135, 1962.
2. McLean, John D.; Schmidt, Stanley F.; and McGee, Leonard A.: Optimal Filtering and Linear Prediction Applied to a Midcourse Navigation System for the Circumlunar Mission. NASA TN D-1208, 1962.
3. Smith, Gerald L.; and Harper, Eleanor V.: Midcourse Guidance Using Radar Tracking and On-Board Observation Data. NASA TN D-2238, 1964.
4. Battin, R. H.: A Statistical Optimizing Navigation Procedure for Space Flight. ARS J., vol. 32, no. 11, Nov. 1962, pp. 1681-1696.
5. White, John S.; Callas, George P.; and Cicolani, Luigi S.: Application of Statistical Filter Theory to the Interplanetary Navigation and Guidance Problem. NASA TN D-2697, 1965.
6. Battin, R. H.: Astronautical Guidance. McGraw-Hill Book Co., Inc., 1964.
7. Smith, Gerald L.: Secondary Errors and Off-Design Conditions in Optimal Estimation of Space Vehicle Trajectories. NASA TN D-2129, 1964.
8. McGee, Leonard A.: Effect of Reduced Computer Precision on a Midcourse Navigation and Guidance System Using Optimal Filtering and Linear Prediction. NASA TN D-3382, 1966.

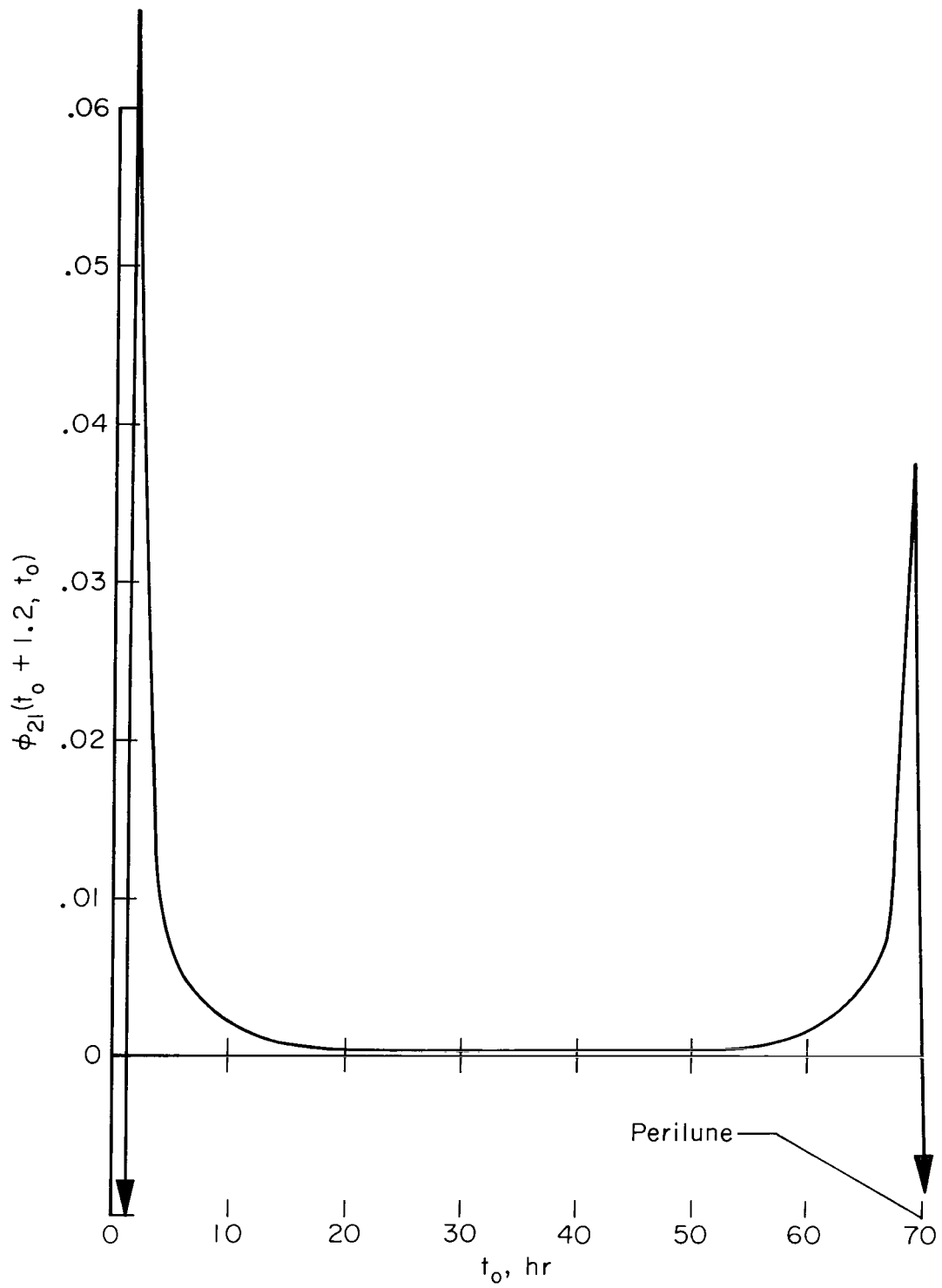
TABLE I.- THE NORM OF THE COMPONENTS OF F VERSUS LOCATION  
IN CISLUNAR SPACE

| Vehicle location          | $\ F_e\ $    | $\ F_m\ $       | $\ F_s\ $       | $\ F_J\ $       |
|---------------------------|--------------|-----------------|-----------------|-----------------|
| Perigee                   | $2.44(10)^4$ | $1.41(10)^{-3}$ | $6.21(10)^{-4}$ | $1.57(10)^2$    |
| 66,000 km from earth      | 2.20         | $2.36(10)^{-3}$ | $6.21(10)^{-4}$ | $1.52(10)^{-3}$ |
| Lunar sphere of influence | .193         | .270            | $6.21(10)^{-4}$ | $4.65(10)^{-7}$ |
| Perilune                  | .115         | $1.49(10)^4$    | $6.21(10)^{-4}$ | $1.98(10)^{-7}$ |



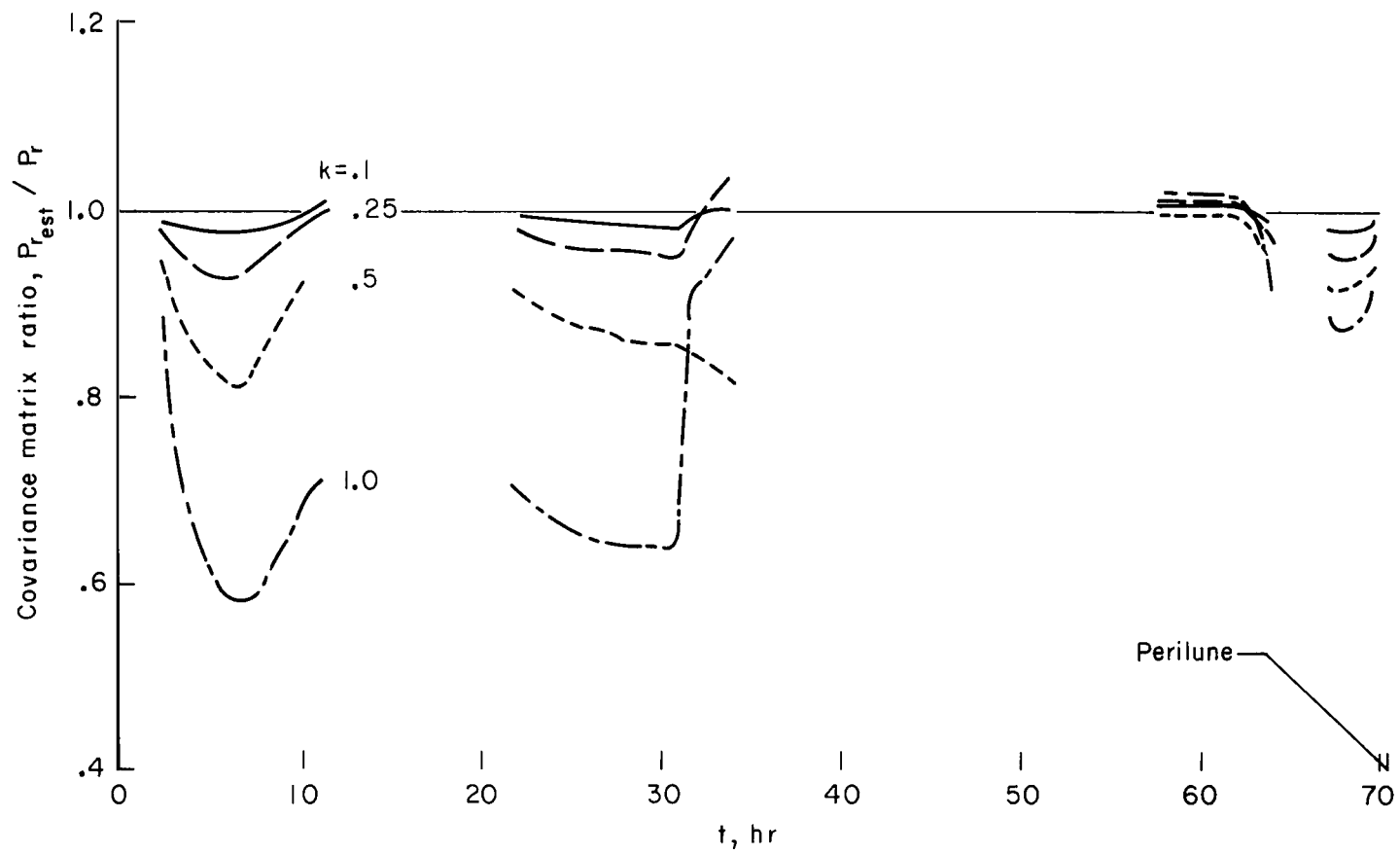
(a) Typical diagonal element.

Figure 1.- Typical elements of the transition matrix versus initial time with final time 1.2 hours later.



(b) Typical off diagonal element.

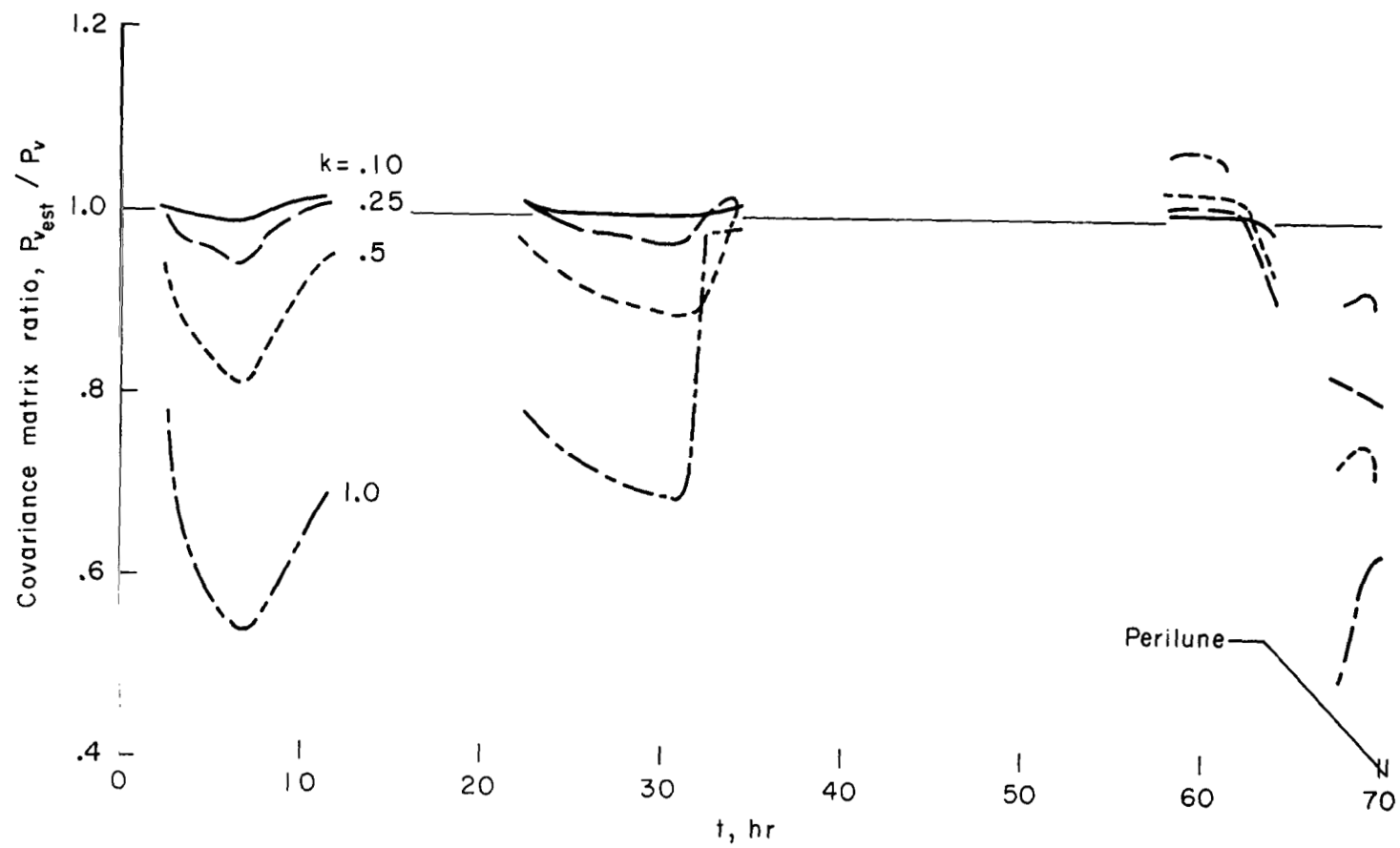
Figure 1.- Concluded.



(a) Position, outbound leg.

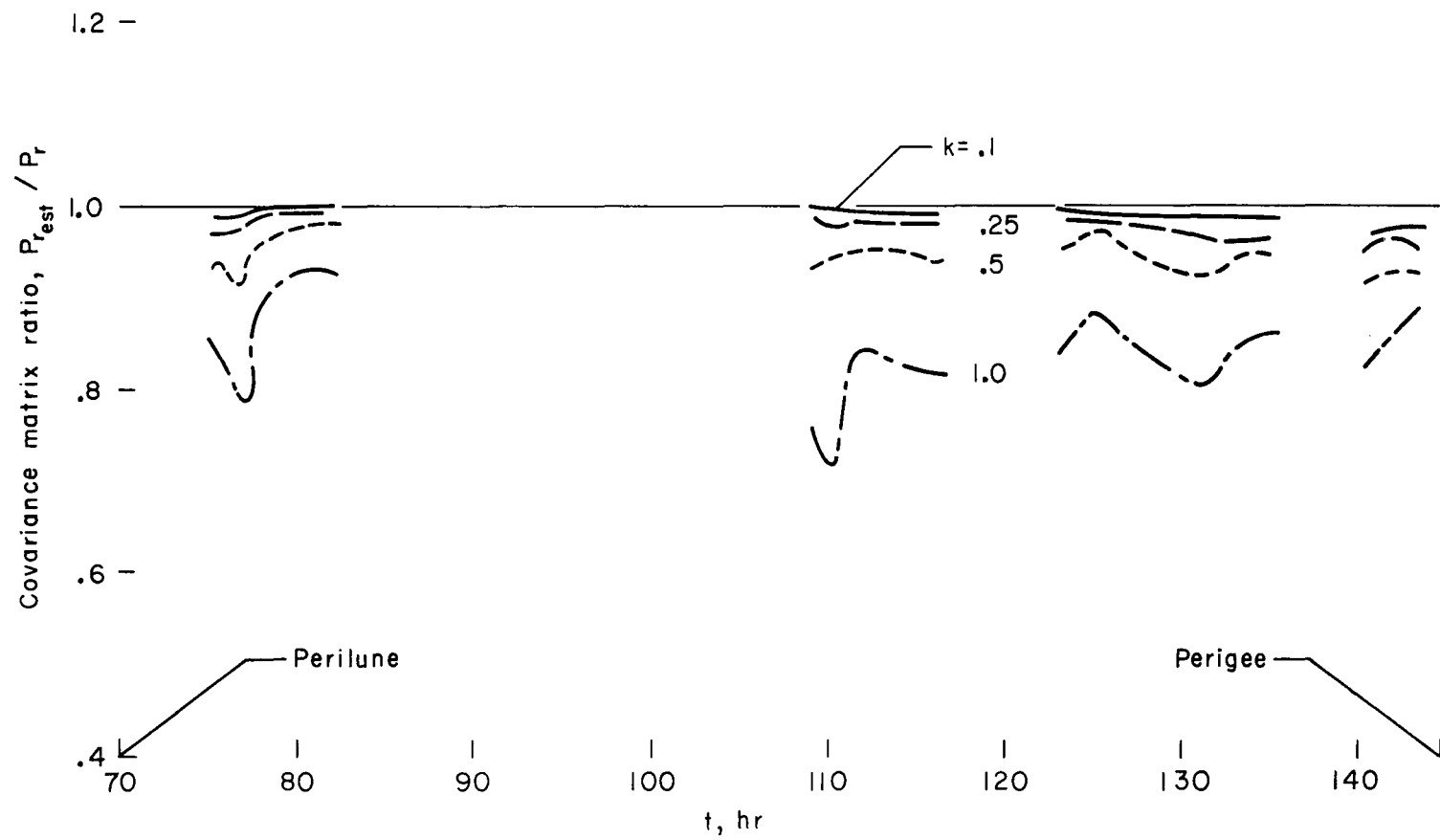
Figure 2.- Comparison of estimated and actual covariance matrices for various values of  $k$ .





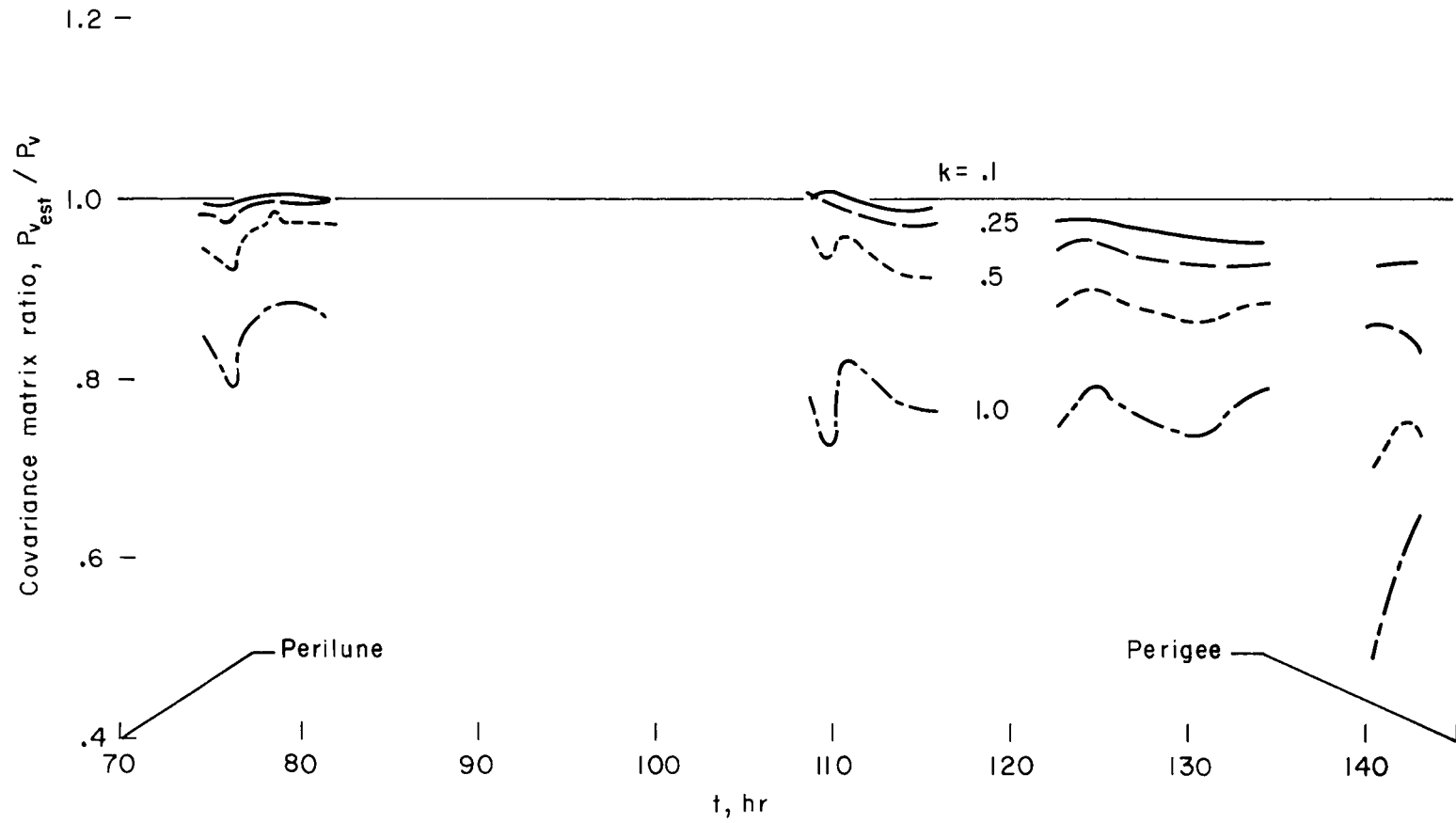
(b) Velocity, outbound leg.

Figure 2.- Continued.



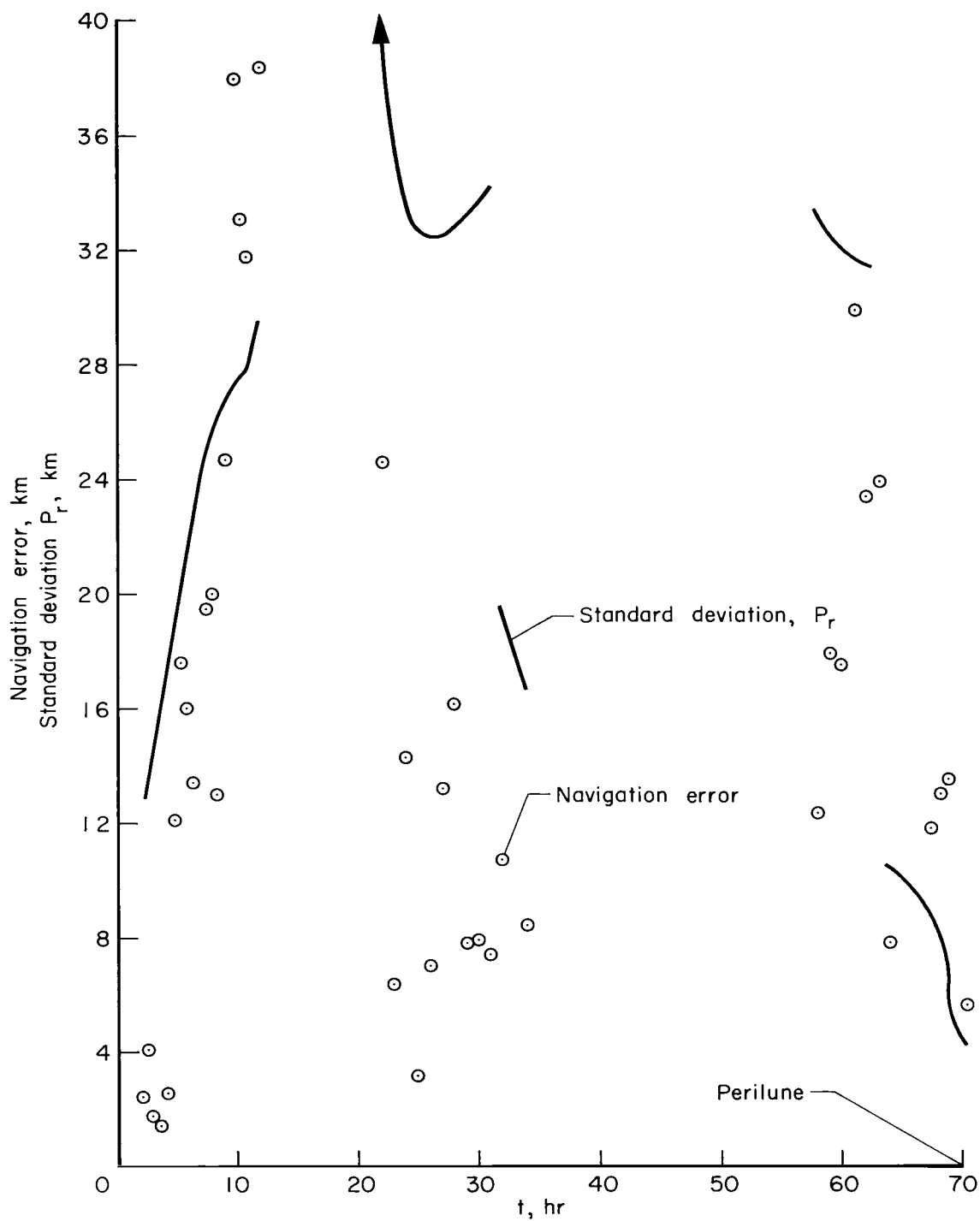
(c) Position, return leg.

Figure 2.- Continued.



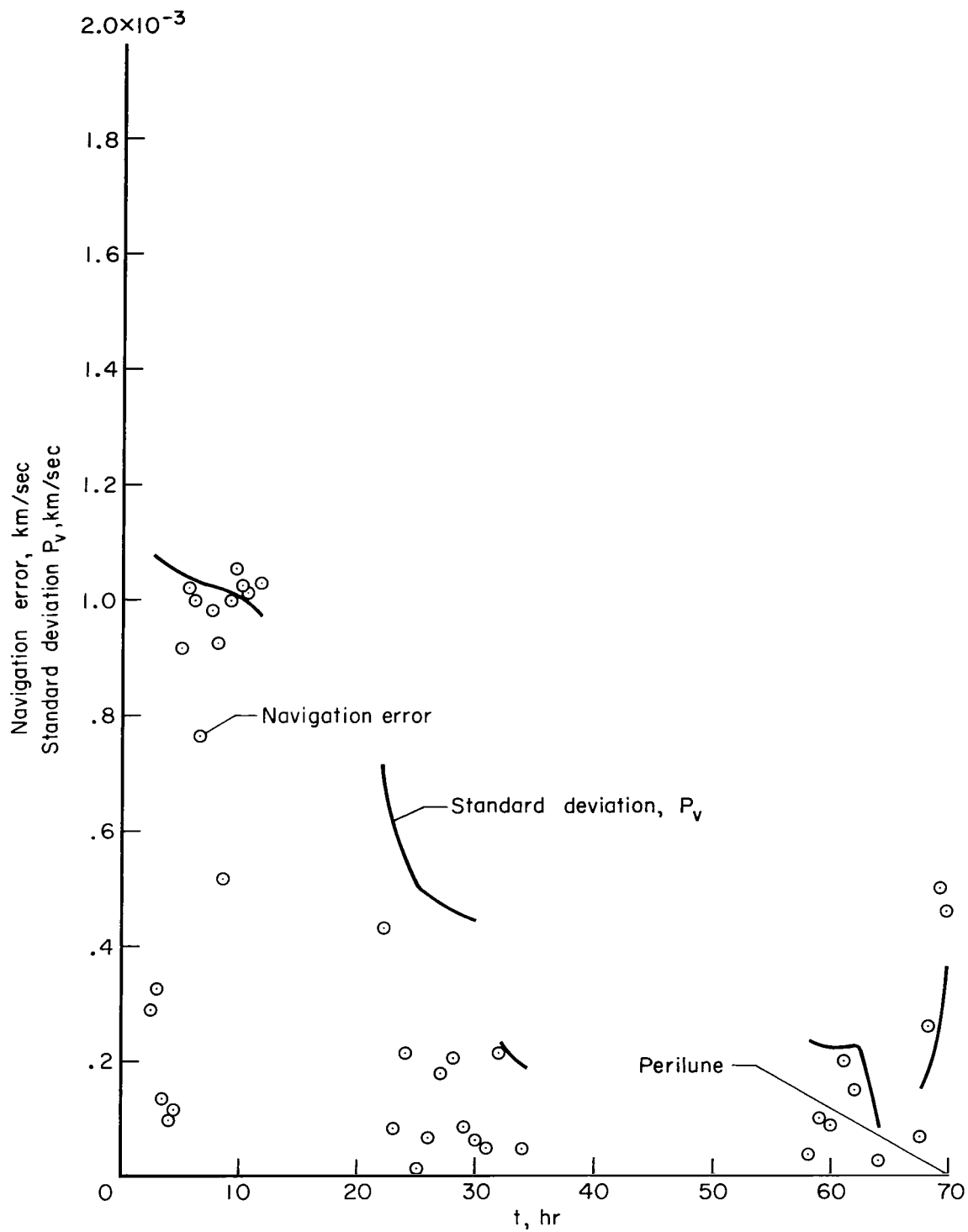
(d) Velocity, return leg.

Figure 2.- Concluded.



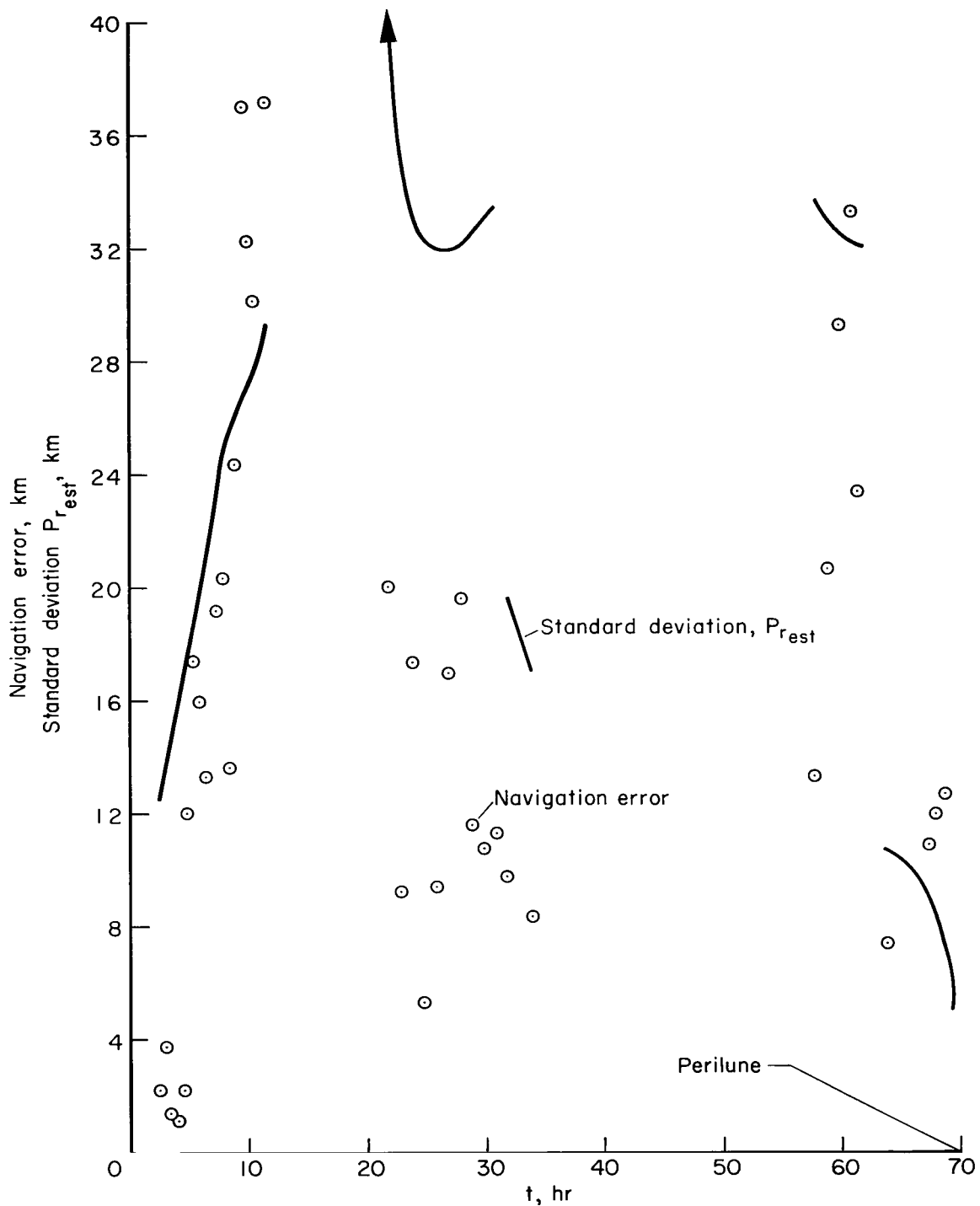
(a) Position.

Figure 3.- Navigation error and its standard deviation for the optimum case.



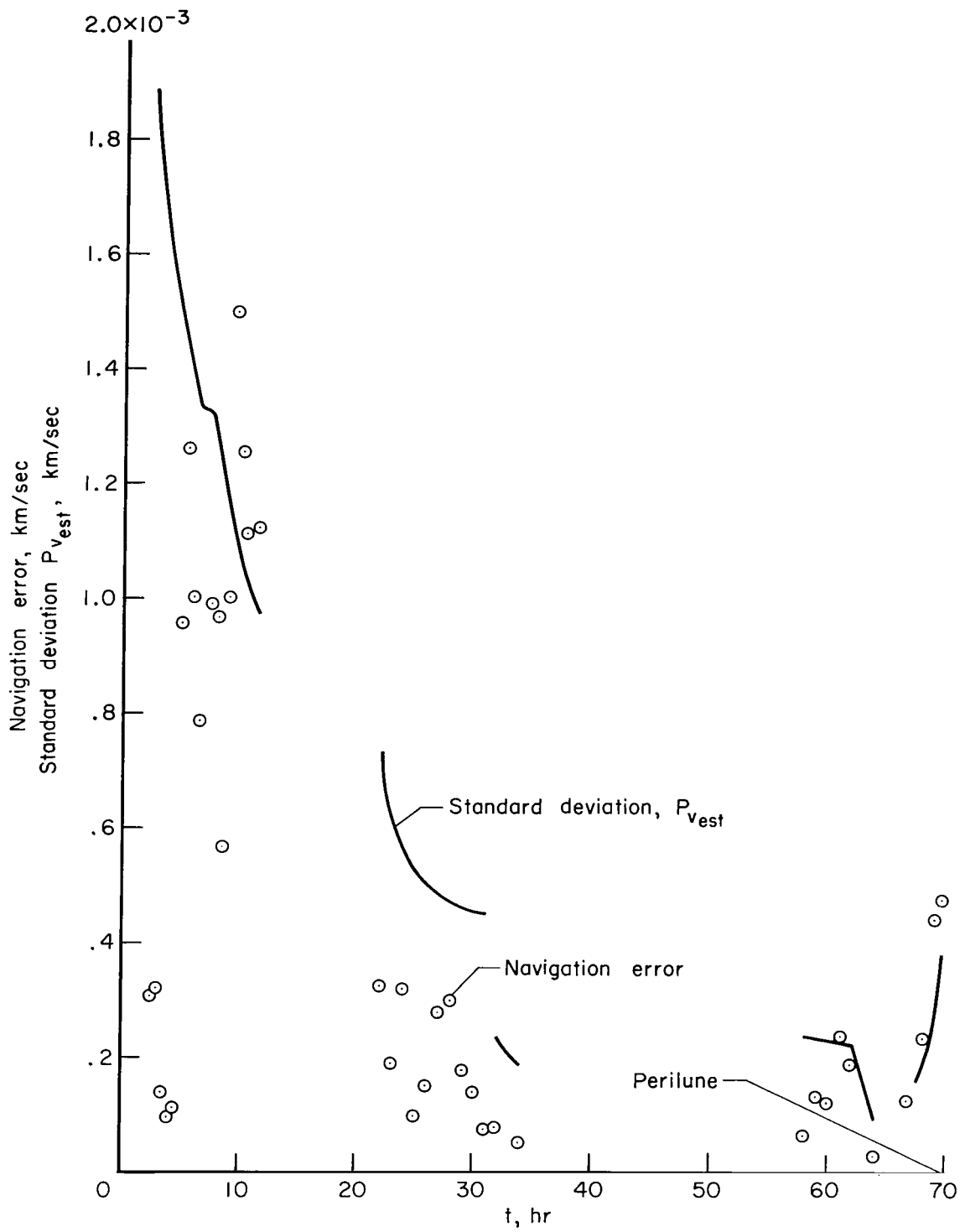
(b) Velocity.

Figure 3.- Concluded.



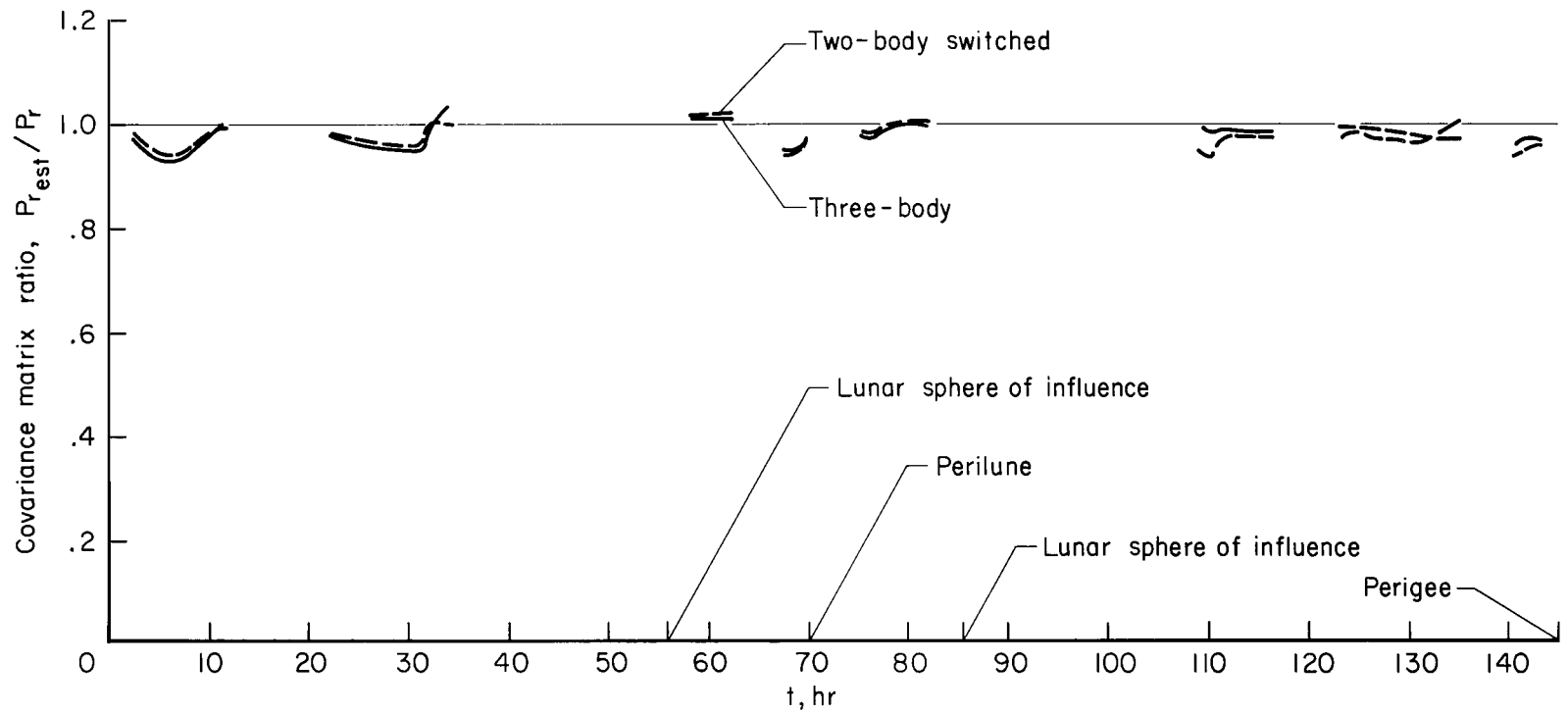
(a) Position.

Figure 4.- Navigation error and its standard deviation using approximate transition matrices,  $k = 0.25$ .



(b) Velocity.

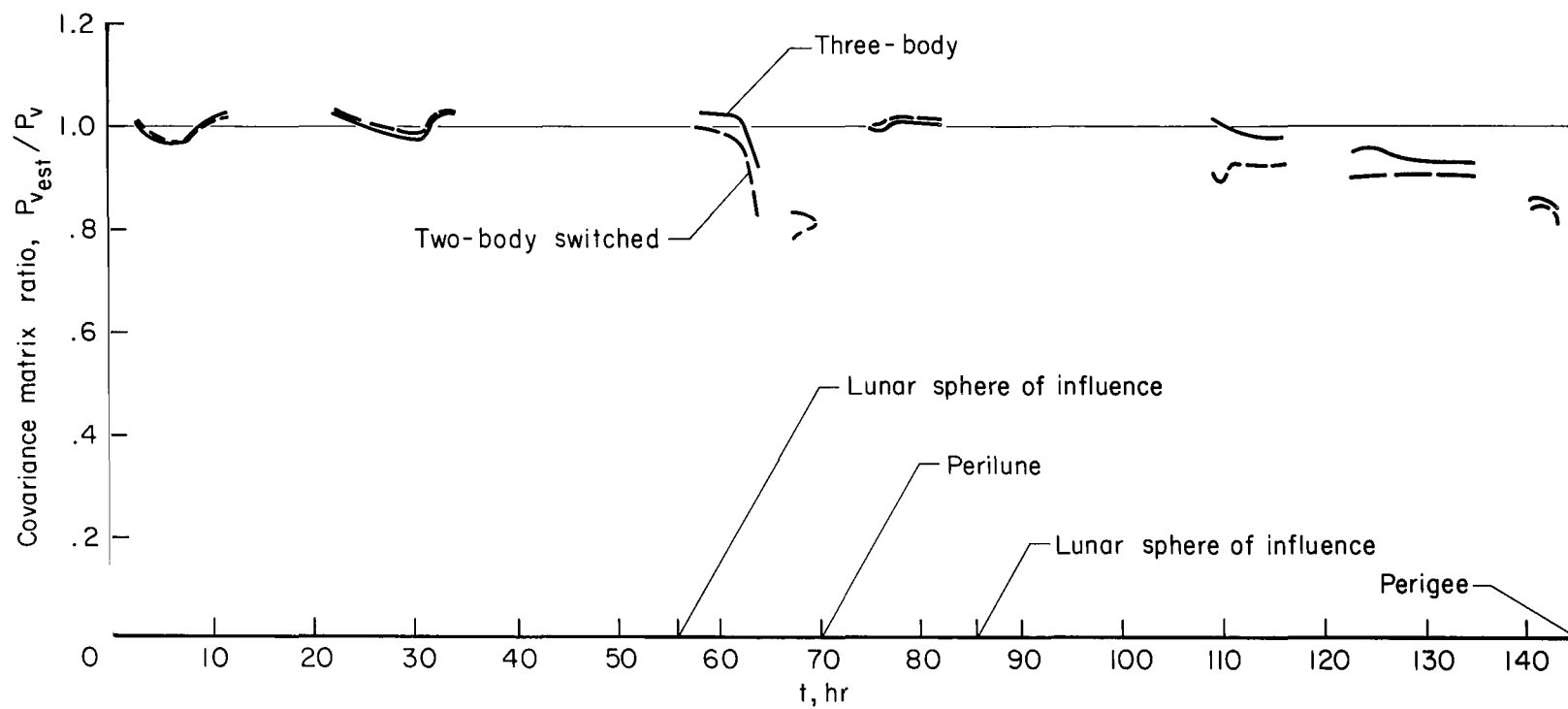
Figure 4.- Concluded.



(a) Position.

Figure 5.- Effect of two-body switched or three-body computation of transition matrices.





(b) Velocity.

Figure 5.- Concluded.

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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